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Infrared behavior of interacting bosons at zero temperature

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We review the infrared behavior of interacting bosons at zero temperature. After a brief discussion of the Bogoliubov approximation and the breakdown of perturbation theory due to infrared divergences, we show how the non-perturbative renormalization group enables to obtain the exact infrared behavior of the correlation functions.

Keywords: Boson systems; Renormalization group methods.

1. Introduction

Many of the predictions of the Bogoliubov theory of superfluidity[1] have been confirmed experimentally, in particular in ultracold atomic gases.[2,3] Nevertheless a clear understanding of the infrared behavior of interacting bosons at zero temperature has remained a challenging theoretical issue for a long time. Early attempts to go beyond the Bogoliubov theory have revealed a singular perturbation theory plagued by infrared divergences due to the presence of the Bose-Einstein condensate and the Goldstone mode.[4,5,6] In the 1970s, Nepomnyashchii and Nepomnyashchii proved that the anomalous self-energy vanishes at zero frequency and momentum in dimension $d \leq 3$.[7] This exact result shows that the Bogoliubov approximation, where the linear spectrum and the superfluidity rely on a finite value of the anomalous self-energy, breaks down at low energy. As realized latter on,[8,9] the singular perturbation theory is a direct consequence of the coupling between transverse and longitudinal fluctuations and reflects the divergence of the longitudinal susceptibility – a general phenomenon in systems with a continuous broken symmetry.[10]

In this paper, we review the infrared behavior of interacting bosons. A more detailed discussion together with a comparison to the classical O(N) model can be found in Ref. [11]. In Sec. 2, we briefly review the Bogoliubov theory and the appearance of infrared divergences in perturbation theory. We introduce the Ginzburg momentum scale p_G signaling the breakdown of the Bogoliubov approximation. In Sec. 3, we discuss the non-perturbative renormalization group (NPRG) and show how it enables to obtain the exact infrared behavior of the

normal and anomalous single-particle propagators without encountering infrared divergences.[12,13,14,15,16,17,18,19,20]

2. Perturbation theory and breakdown of the Bogoliubov approximation

We consider interacting bosons at zero temperature with the (Euclidean) action

$$S = \int dx \left[\psi^* \left(\partial_\tau - \mu - \frac{\boldsymbol{\nabla}^2}{2m} \right) \psi + \frac{g}{2} (\psi^* \psi)^2 \right], \tag{1}$$

where $\psi(x)$ is a bosonic (complex) field, $x = (\mathbf{r}, \tau)$, and $\int dx = \int_0^\beta d\tau \int d^d r$. $\tau \in [0, \beta]$ is an imaginary time, $\beta \to \infty$ the inverse temperature, and μ denotes the chemical potential. The interaction is assumed to be local in space and the model is regularized by a momentum cutoff Λ. We consider a space dimension $d > 1$.

Introducing the two-component field

$$\Psi(p) = \begin{pmatrix} \psi(p) \\ \psi^*(-p) \end{pmatrix}, \quad \Psi^\dagger(p) = \big(\psi^*(p), \psi(-p) \big) \tag{2}$$

(with $p = (\mathbf{p}, i\omega)$ and ω a Matsubara frequency), the one-particle (connected) propagator becomes a 2×2 matrix whose inverse in Fourier space is given by

$$\begin{pmatrix} i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{\rm n}(p) & -\Sigma_{\rm an}(p) \\ -\Sigma^*_{\rm an}(p) & -i\omega + \mu - \epsilon_{\mathbf{p}} - \Sigma_{\rm n}(-p) \end{pmatrix}, \tag{3}$$

where $\Sigma_{\rm n}$ and $\Sigma_{\rm an}$ are the normal and anomalous self-energies, respectively, and $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. If we choose the order parameter $\langle \psi(x) \rangle = \sqrt{n_0}$ to be real (with n_0 the condensate density), then the anomalous self-energy $\Sigma_{\rm an}(p)$ is real.

2.1. *Bogoliubov approximation*

The Bogoliubov approximation is a Gaussian fluctuation theory about the saddle point solution $\psi(x) = \sqrt{n_0} = \sqrt{\mu/g}$. It is equivalent to a zero-loop calculation of the self-energies,[21,22]

$$\Sigma^{(0)}_{\rm n}(p) = 2gn_0, \quad \Sigma^{(0)}_{\rm an}(p) = gn_0. \tag{4}$$

This yields the (connected) propagators

$$\begin{aligned} G^{(0)}_{\rm n}(p) &= -\langle \psi(p) \psi^*(p) \rangle_c = \frac{-i\omega - \epsilon_{\mathbf{p}} - gn_0}{\omega^2 + E^2_{\mathbf{p}}}, \\ G^{(0)}_{\rm an}(p) &= -\langle \psi(p) \psi(-p) \rangle_c = \frac{gn_0}{\omega^2 + E^2_{\mathbf{p}}}, \end{aligned} \tag{5}$$

where $E_{\mathbf{p}} = [\epsilon_{\mathbf{p}}(\epsilon_{\mathbf{p}} + 2gn_0)]^{1/2}$ is the Bogoliubov quasi-particle excitation energy. When $|\mathbf{p}|$ is larger than the healing momentum $p_c = (2gmn_0)^{1/2}$, the spectrum $E_{\mathbf{p}} \simeq \epsilon_{\mathbf{p}} + gn_0$ is particle-like, whereas it becomes sound-like for $|\mathbf{p}| \ll p_c$ with a velocity $c = \sqrt{gn_0/m}$. In the weak-coupling limit, $n_0 \simeq \bar{n}$ ($\bar{n}$ is the mean boson

density) and p_c can equivalently be defined as $p_c = (2gm\bar{n})^{1/2}$. In the Bogoliubov approximation, the occurrence of a linear spectrum at low energy (which implies superfluidity according to Landau's criterion) is due to $\Sigma_{\rm an}(p=0)$ being nonzero.

2.2. *Infrared divergences and the Ginzburg scale*

The lowest-order (one-loop) correction $\Sigma^{(1)}$ to the Bogoliubov result $\Sigma^{(0)}$ is divergent for $d \leq 3$. Retaining only the divergent part, we obtain

$$\Sigma^{(1)}_{\rm n}(p) \simeq \Sigma^{(1)}_{\rm an}(p) \simeq -2\frac{g^4 n_0^3}{c^3} A_{d+1} \left(\mathbf{p}^2 + \frac{\omega^2}{c^2} \right)^{(d-3)/2} \tag{6}$$

if $d < 3$ and

$$\Sigma^{(1)}_{\rm n}(p) \simeq \Sigma^{(1)}_{\rm an}(p) \simeq -\frac{g^4 n_0^3}{c^3} A_4 \ln\left(\frac{p_c^2}{\mathbf{p}^2 + \omega^2/c^2} \right) \tag{7}$$

if $d = 3$, where

$$A_d = \begin{cases} -\frac{2^{1-d}\pi^{1-d/2}}{\sin(\pi d/2)} \frac{\Gamma(d/2)}{\Gamma(d-1)} & \text{if } d < 4, \\ \frac{1}{8\pi^2} & \text{if } d = 4. \end{cases} \tag{8}$$

We can estimate the characteristic (Ginzburg) momentum scale p_G below which the Bogoliubov approximation breaks down from the condition $|\Sigma^{(1)}_{\rm n}(p)| \sim \Sigma^{(0)}_{\rm n}(p)$ or $|\Sigma^{(1)}_{\rm an}(p)| \sim \Sigma^{(0)}_{\rm an}(p)$ for $|\mathbf{p}| = p_G$ and $|\omega| = cp_G$,

$$p_G \sim \begin{cases} (A_{d+1} gmp_c)^{1/(3-d)} & \text{if } d < 3, \\ p_c \exp\left(-\frac{1}{A_4 gmp_c} \right) & \text{if } d = 3. \end{cases} \tag{9}$$

This result can be rewritten as

$$p_G \sim \begin{cases} p_c (A_{d+1} \tilde{g}^{d/2})^{1/(3-d)} & \text{if } d < 3, \\ p_c \exp\left(-\frac{1}{A_4 \sqrt{2} \tilde{g}^{3/2}} \right) & \text{if } d = 3, \end{cases} \tag{10}$$

where

$$\tilde{g} = gm\bar{n}^{1-2/d} \sim \left(\frac{p_c}{\bar{n}^{1/d}} \right)^2 \tag{11}$$

is the dimensionless coupling constant obtained by comparing the mean interaction energy per particle $g\bar{n}$ to the typical kinetic energy $1/m\bar{r}^2$ where $\bar{r} \sim \bar{n}^{-1/d}$ is the mean distance between particles.[23] A superfluid is weakly correlated if $\tilde{g} \ll 1$, i.e. $p_G \ll p_c \ll \bar{n}^{1/d}$ (the characteristic momentum scale $\bar{n}^{1/d}$ does however not play any role in the weak-coupling limit). In this case, the Bogoliubov theory applies to a large part of the spectrum where the dispersion is linear (i.e. $|\mathbf{p}| \lesssim p_c$) and breaks down only at very small momenta $|\mathbf{p}| \lesssim p_G \ll p_c$. When the dimensionless coupling $\tilde{g}$ becomes of order unity, the three characteristic momentum scales $p_G \sim p_c \sim \bar{n}^{1/d}$ become of the same order. The momentum range $[p_G, p_c]$ where the linear spectrum can be described by the Bogoliubov theory is then suppressed. We expect the strong-coupling regime $\tilde{g} \gg 1$ to be governed by a single characteristic momentum scale, namely $\bar{n}^{1/d}$.[24]

2.3. *Vanishing of the anomalous self-energy*

The exact values of $\Sigma_{\rm n}(p=0)$ and $\Sigma_{\rm an}(p=0)$ can be obtained using the U(1) symmetry of the action, i.e. the invariance under the field transformation $\psi(x) \to e^{i\theta}\psi(x)$ and $\psi^*(x) \to e^{-i\theta}\psi^*(x)$. On the one hand, the self-energies satisfy the Hugenholtz-Pines theorem,[4]

$$\Sigma_{\rm n}(p=0) - \Sigma_{\rm an}(p=0) = \mu. \tag{12}$$

On the other hand, the anomalous self-energy vanishes,

$$\Sigma_{\rm an}(p=0) = 0. \tag{13}$$

The last result was first proven by Nepomnyashchii and Nepomnyashchii.[7,20,11] It shows that the Bogoliubov theory, where the linear spectrum and the superfluidity rely on a finite value of the anomalous self-energy, breaks down at low energy in agreement with the conclusions drawn from perturbation theory (Sec. 2.2).

3. The non-perturbative RG

The NPRG enables to circumvent the difficulties of perturbation theory and derive the correlation functions in the low-energy limit.[12,13,16,17,18,14,15,19,20] The strategy of the NPRG is to build a family of theories indexed by a momentum scale k such that fluctuations are smoothly taken into account as k is lowered from the microscopic scale Λ down to 0.[25,26] This is achieved by adding to the action (1) an infrared regulator term

$$\Delta S_k[\psi^*,\psi] = \sum_p \psi^*(p) R_k(p) \psi(p). \tag{14}$$

The main quantity of interest is the so-called average effective action

$$\Gamma_k[\phi^*,\phi] = -\ln Z_k[J^*,J] + \sum_p [J^*(p)\phi(p) + \text{c.c.}] - \Delta S_k[\phi^*,\phi], \tag{15}$$

defined as a modified Legendre transform of $-\ln Z_k[J^*,J]$ which includes the subtraction of $\Delta S_k[\phi^*,\phi]$. J denotes a complex external source that couples linearly to the boson field ψ and $\phi(x) = \langle \psi(x) \rangle$ is the superfluid order parameter. The cutoff function R_k is chosen such that at the microscopic scale Λ it suppresses all fluctuations, so that the mean-field approximation $\Gamma_\Lambda[\phi^*,\phi] = S[\phi^*,\phi]$ becomes exact. The effective action of the original model (1) is given by $\Gamma_{k=0}$ provided that $R_{k=0}$ vanishes. For a generic value of k, the cutoff function $R_k(p)$ suppresses fluctuations with momentum $|\mathbf{p}| \lesssim k$ and frequency $|\omega| \lesssim ck$ but leaves those with $|\mathbf{p}|, |\omega|/c \gtrsim k$ unaffected ($c \equiv c_k$ is the velocity of the Goldstone mode). The dependence of the average effective action on k is given by Wetterich's equation[28]

$$\partial_t \Gamma_k[\phi^*,\phi] = \frac{1}{2} \mathrm{Tr} \left\{ \dot{R}_k \left(\Gamma_k^{(2)}[\phi^*,\phi] + R_k \right)^{-1} \right\}, \tag{16}$$

where $t = \ln(k/\Lambda)$ and $\dot{R}_k = \partial_t R_k$. $\Gamma_k^{(2)}[\phi^*, \phi]$ denotes the second-order functional derivative of $\Gamma_k[\phi]$. In Fourier space, the trace involves a sum over momenta and frequencies as well as the internal index of the ϕ field.

3.1. *Derivative expansion and infrared behavior*

Because of the regulator term ΔS_k, the vertices $\Gamma_k^{(n)}(p_1, \cdots, p_n)$ are smooth functions of momenta and frequencies and can be expanded in powers of $\mathbf{p}_i^2/k^2$ and ω_i^2/c^2k^2. Thus if we are interested only in the low-energy properties, we can use a derivative expansion of the average effective action.[25,26] In the following we consider the ansatz[27]

$$\Gamma_k[\phi^*, \phi] = \int dx \Big[\phi^* \Big(Z_{C,k}\partial_\tau - V_{A,k}\partial_\tau^2 - \frac{Z_{A,k}}{2m}\boldsymbol{\nabla}^2 \Big)\phi + \frac{\lambda_k}{2}(n - n_{0,k})^2\Big], \tag{17}$$

where $n = |\phi|^2$. $n_{0,k}$ denotes the condensate density in the equilibrium state. We have introduced a second-order time derivative term. Although not present in the initial average effective action Γ_Λ, we shall see that this term plays a crucial role when $d \leq 3$.[14,16]

In a broken U(1) symmetry state with real order parameter $\phi = \sqrt{n_0}$, the normal and anomalous self-energies are given by

$$\begin{aligned} \Sigma_{k,\mathrm{n}}(p) &= \mu + V_{A,k}\omega^2 + (1 - Z_{C,k})i\omega - (1 - Z_{A,k})\epsilon_{\mathbf{p}} + \lambda_k n_{0,k}, \\ \Sigma_{k,\mathrm{an}}(p) &= \lambda_k n_{0,k}. \end{aligned} \tag{18}$$

These expressions imply the existence of a sound mode with velocity

$$c_k = \left(\frac{Z_{A,k}/2m}{V_{A,k} + Z_{C,k}^2/2\lambda_k n_{0,k}} \right)^{1/2}. \tag{19}$$

At the initial stage of the flow, $Z_{A,\Lambda} = Z_{C,\Lambda} = 1$, $V_{A,\Lambda} = 0$, $\lambda_\Lambda = g$ and $n_{0,\Lambda} = \mu/g$, which reproduces the results of the Bogoliubov approximation. A crucial property of the RG flow is that

$$\lambda_k \sim k^{3-d} \tag{20}$$

vanishes with k when $d \leq 3$. Eq. (20) follows from the numerical solution of the RG equations, but can also be anticipated from the expected singular behavior of the longitudinal propagator.[11]

The parameters $Z_{A,k}$, $Z_{C,k}$ and $V_{A,k}$ can be related to thermodynamic quantities using Ward identities,[5,13,18]

$$\begin{aligned} n_{s,k} &= Z_{A,k} n_{0,k} = \bar{n}_k, \\ V_{A,k} &= -\frac{1}{2n_{0,k}} \frac{\partial^2 U_k}{\partial \mu^2}\bigg|_{n_{0,k}}, \\ Z_{C,k} &= -\frac{\partial^2 U_k}{\partial n \partial \mu}\bigg|_{n_{0,k}} = \lambda_k \frac{dn_{0,k}}{d\mu}, \end{aligned} \tag{21}$$

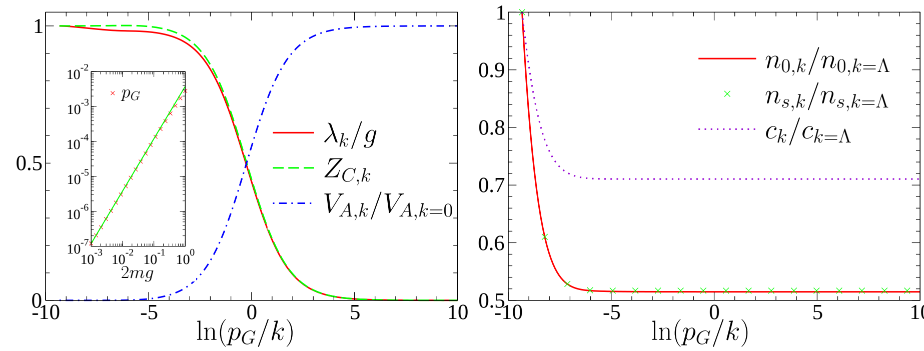

Fig. 1. (Color online) (Left panel) λ_k, $Z_{C,k}$ and $V_{A,k}$ vs $\ln(p_G/k)$ where $p_G = \sqrt{(gm)^3\bar{n}}/4\pi$ for $\bar{n} = 0.01$, $2mg = 0.1$ and $d = 2$ [$\ln(p_G/p_c) \simeq -5.87$]. The inset shows p_G vs $2mg$ obtained from the criterion $V_{A,p_G} = V_{A,k=0}/2$ [the Green solid line is a fit to $p_G \sim (2mg)^{3/2}$]. (Right panel) Condensate density $n_{0,k}$, superfluid density $n_{s,k}$ and Goldstone mode velocity c_k vs $\ln(p_G/k)$.

where $\bar{n}_k$ is the mean boson density and $n_{s,k}$ the superfluid density. Here we consider the effective potential U_k as a function of the two independent variables n and μ. The first of equations (21) states that in a Galilean invariant superfluid at zero temperature, the superfluid density is given by the full density of the fluid.[5] Equations (21) also imply that the Goldstone mode velocity c_k coincides with the macroscopic sound velocity,[5,13,18] i.e.

$$\frac{d\bar{n}_k}{d\mu} = \frac{\bar{n}_k}{mc_k^2}. \tag{22}$$

Since thermodynamic quantities, including the condensate "compressibility" $dn_{0,k}/d\mu$ should remain finite in the limit $k \to 0$, we deduce from (21) that $Z_{C,k} \sim \lambda_k \sim k^{3-d}$ vanishes in the infrared limit, and

$$\lim_{k\to 0} c_k = \lim_{k\to 0} \left(\frac{Z_{A,k}}{2mV_{A,k}} \right)^{1/2} . \tag{23}$$

Both $Z_{A,k} = \bar{n}_k/n_{0,k}$ and the macroscopic sound velocity c_k being finite at $k = 0$, $V_{A,k}$ (which vanishes in the Bogoliubov approximation) takes a non-zero value when $k \to 0$.

The suppression of $Z_{C,k}$, together with a finite value of $V_{A,k=0}$ shows that the average effective action (17) exhibits a "relativistic" invariance in the infrared limit and therefore becomes equivalent to that of the classical O(2) model in dimensions $d+1$. In the ordered phase, the coupling constant of this model vanishes as $\lambda_k \sim k^{4-(d+1)}$,[11] which agrees with (20). For $k \to 0$, the existence of a linear spectrum is due to the relativistic form of the average effective action (rather than a non-zero value of $\lambda_k n_{0,k}$ as in the Bogoliubov approximation).

To obtain the $k = 0$ limit of the propagators (at fixed p), one should in principle stop the flow when $k \sim \sqrt{\mathbf{p}^2 + \omega^2/c^2}$.[18] Since thermodynamic quantities are not expected to flow in the infrared limit, they can be approximated by their $k =$

0 values. Making use of the Ward identities (21), we deduce the exact infrared behavior of the normal and anomalous propagators (at $k = 0$),[18]

$$
\begin{aligned}
G_{\rm n}(p) &= -\frac{n_0 mc^2}{\bar{n}} \frac{1}{\omega^2 + c^2\mathbf{p}^2} - \frac{mc^2}{\bar{n}} \frac{dn_0}{d\mu} \frac{i\omega}{\omega^2 + c^2\mathbf{p}^2} - \frac{1}{2} G_\parallel(p), \\
G_{\rm an}(p) &= \frac{n_0 mc^2}{\bar{n}} \frac{1}{\omega^2 + c^2\mathbf{p}^2} - \frac{1}{2} G_\parallel(p),
\end{aligned} \tag{24}
$$

where

$$
G_\parallel(p) = \frac{1}{2n_0 C(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}} \tag{25}
$$

is the propagator of the longitudinal fluctuations. The constant C follows from the replacement $\lambda_k \to C(\omega^2 + c^2\mathbf{p}^2)^{(3-d)/2}$. The leading terms in (24) agree with the results of Gavoret and Nozières.[5] The contribution of the diverging longitudinal correlation function was first identified by Nepomnyashchii and Nepomnyashchii.[8,9]

3.2. *RG flows*

The conclusions of the preceding section can be obtained more rigorously from the RG equation (16) satisfied by the average effective action. The flow of λ_k, $Z_{C,k}$ and $V_{A,k}$ is shown in Fig. 1 for a two-dimensional system in the weak-coupling limit. We clearly see that the Bogoliubov approximation breaks down at a characteristic momentum scale $p_G \sim \sqrt{(gm)^3\bar{n}}$. In the Goldstone regime $k \ll p_G$, we find that both λ_k and $Z_{C,k}$ vanish linearly with k in agreement with the conclusions of Sec. 3.1. Furthermore, $V_{A,k}$ takes a finite value in the limit $k \to 0$ in agreement with the limiting value (23) of the Goldstone mode velocity. Figure 1 also shows the behavior of the condensate density $n_{0,k}$, the superfluid density $n_{s,k} = Z_{A,k} n_{0,k}$ and the velocity c_k. Since $Z_{A,k=0} \simeq 1.004$, the mean boson density $\bar{n}_k = n_{s,k}$ is nearly equal to the condensate density $n_{0,k}$. Apart from a slight variation at the beginning of the flow, $n_{0,k}$, $n_{s,k} = Z_{A,k} n_{0,k}$ and c_k do not change with k. In particular, they are not sensitive to the Ginzburg scale p_G. This result is quite remarkable for the Goldstone mode velocity c_k, whose expression (19) involves the parameters λ_k, $Z_{C,k}$ and $V_{A,k}$, which all strongly vary when $k \sim p_G$. These findings are a nice illustration of the fact that the divergence of the longitudinal susceptibility does not affect local gauge invariant quantities.[13,18]

4. Conclusion

Interacting bosons at zero temperature are characterized by two momentum scales: the healing (or hydrodynamic) scale p_c and the Ginzburg scale p_G. p_G sets the scale at which the Bogoliubov approximation breaks down. For momenta $|\mathbf{p}| \ll p_c$, it is possible to use a hydrodynamic description in terms of density and phase variables. This description allows one to compute the correlation functions without encountering infrared divergences.[29,11] In this paper, we have reviewed another

approach, based on the NPRG, which enables to describe the system at all energy scales and yields the exact infrared behavior of the single-particle propagator. A nice feature of the NPRG is that it can be used to study models of strongly-correlated bosons such as the Bose-Hubbard model.[30]

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